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Reduction theory for a rational function field

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Abstract. Let G be a split reductive group over a finite field $\mathbf{F}_q$. Let $F = \mathbf{F}_q(t)$ and let $\mathbf{A}$ denote the adèles of F. We show that every double coset in $G(F)\backslash G(\mathbf{A})/K$ has a representative in a maximal split torus of G. Here K is the set of integral adèlic points of G. When G ranges over general linear groups this is equivalent to the assertion that any algebraic vector bundle over the projective line is isomorphic to a direct sum of line bundles.

Keywords. Automorphic form; function field.

1. Introduction

Let F be a global field, $\mathbf{A}$ its ring of adèles and G a reductive group defined over F. The theory of automorphic forms involves the study of spaces of functions on $G(F)\backslash G(\mathbf{A})$ as representations of $G(\mathbf{A})$. The functions involved are often required to be right invariant under certain large compact subgroups K of $G(\mathbf{A})$ because (among other reasons) the double coset space $G(F)\backslash G(\mathbf{A})/K$ admits nice interpretations. For example, the classical study of the upper half plane modulo the action of arithmetic subgroups of the real special linear group is a special case of the above when F is the field of rational numbers (see e.g., ([13], §1). Another special case, which corresponds to taking F to be a field of rational functions in one variable and G to be $GL(2)$ is discussed by Weil in [15]. When F is a function field, Harder describes a fundamental domain for the action of $G(F)$ on $G(\mathbf{A})$ in ([10], §1) using results from [8] and [9]. This is an analogue of the Seigel domain described by Godement in [6] for $F = \mathbf{Q}$. Proposition 14 in this article is analogous to these results and the proof proceeds along the lines of [6]. Harder's description of the fundamental domain is a very basic result in the theory of automorphic forms over function fields (see e.g., [12], §9 and Appendix E).

From now on let G be a split reductive group defined over a finite field $\mathbf{F}_q$ with q elements. Fix a Borel subgroup B defined over $\mathbf{F}_q$ with unipotent radical N, and a maximal $\mathbf{F}_q$-split torus T contained in B. Set $F = \mathbf{F}_q(t)$. For a valuation v of F, we denote the corresponding local field by F_v and its ring of integers by $\mathbf{O}_v$. For each v, fix a uniformizing element $\pi_v \in F \cap \mathbf{O}_v$. In particular, fix $\pi_\infty = t^{-1}$ as a uniformizing element at the place ∞ whose local field is $\mathbf{F}_q((t^{-1}))$. Let K be the maximal compact subgroup $\prod_v G(\mathbf{O}_v)$ of $G(\mathbf{A})$. This article concerns the double coset space

$$G(F)\backslash G(\mathbf{A})/K$$

which may be interpreted as the set of isomorphism classes of principal G-bundles on the projective line. In [7], Grothendieck proves that when G is a complex reductive group any

holomorphic G-bundle over the complex projective line admits a reduction of structure group to a maximal torus. (In fact this result has been attributed to Dedekind and Weber for $G = GL(n)$ by Geyer ([5], §6) who deduces it from a statement in ([3], §22).) In our adèlic setting, this should correspond to the assertion that every double coset has a representative in $T(\mathbf{A})$.

Let $X_*(T)$ denote the lattice $\mathrm{Hom}(\mathbf{G}_m, T)$ of algebraic co-characters of T. Given $\eta \in X_*(T)$, and a valuation v denote by π_v^η the element $\eta(\pi_v) \in T(F_v) \subset T(\mathbf{A})$. Recall that $\eta \in X_*(T)$ is called *antidominant* if $|\alpha_i \circ \eta(\pi_v)|_v \geq 1$ for each simple root α_i (see §3.). Precisely stated, the main result of this article is the following:

Theorem 1. *Every double coset in*

$$G(F)\backslash G(\mathbf{A})/K$$

has a unique representative of the form $(t^{-1})^\eta$, *where* $\eta \in X_*(T)$ *is antidominant.*

In §6., we will deduce Theorem 1 from the following local result which is proved in §5.. Let $F_\bullet$ be the local field $\mathbf{F}_q((\pi))$ of Laurent series in π with coefficients in $\mathbf{F}_q$. It contains, as its ring of integers, the discrete valuation ring $\mathbf{O} = \mathbf{F}_q[[\pi]]$, and as a discrete subring, the polynomial ring $R = \mathbf{F}_q[\pi^{-1}]$. Let $\Gamma = G(R)$.

Theorem 2. *Every double coset in*

$$\Gamma\backslash G(F_\bullet)/G(\mathbf{O})$$

has a unique representative of the form π^η, *where* $\eta \in X_*(T)$ *is antidominant.*

The main results proved in this article should be known to the experts, but we have not found them in the literature beyond the case of $GL(2)$, for which Theorem 2 is proved in ([15], §3). The results proved in this paper have played an important role in the author's work [14], as well as in the work of other authors on $\mathbf{F}_q(t)$ [4,1,11].

2. Normed local vector spaces

Let V be a vector space defined over $\mathbf{F}_q$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a basis of the free $\mathbf{O}$-module $V(\mathbf{O})$ (so that $V(\mathbf{O})$ is isomorphic to the free $\mathbf{O}$-module generated by the $\mathbf{e}_i$s). Given a vector $\mathbf{x} \in V(F_\bullet)$, we may write $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, uniquely, with $x_i \in F_\bullet$. Define

$$\|\mathbf{x}\| = \sup\{|x_1|, \ldots, |x_n|\}. \tag{1}$$

Lemma 4. *If* $g \in GL(V(\mathbf{O}))$, *then* $\|\mathbf{x}g\| = \|\mathbf{x}\|$.

Proof. Let (g_{ij}) be the matrix of G with respect to the basis chosen above. Let $\mathbf{y} = \mathbf{x}g$. If $\mathbf{y} = y_1\mathbf{e}_1 + \cdots + y_n\mathbf{e}_n$, then

$$y_j = \sum_{i=1}^{n} x_i g_{ij}$$

and

$$\begin{aligned}
\|\mathbf{y}\| &= \sup_{1\le j\le n} \Big|\sum_{i=1}^{n} x_i g_{ij}\Big| \\
&\le \sup_{1\le j\le n}\ \sup_{1\le i\le n} |x_i g_{ij}| \qquad \text{(ultrametric inequality)} \\
&\le \sup_{1\le j\le n}\ \sup_{1\le i\le n} |x_i| \qquad \text{(since } g_{ij}\in \mathbf{O}\text{)} \\
&= \|\mathbf{x}\|.
\end{aligned}$$

Hence

$$\|\mathbf{y}\| \le \|\mathbf{x}\|.$$

We may apply the same reasoning to g^{-1} to show that

$$\|\mathbf{x}\| \le \|\mathbf{y}\|.$$

Therefore,

$$\|\mathbf{y}\| = \|\mathbf{x}\|.$$

□

COROLLARY 5.

The norm $\|\cdot\|$ *is independent of our choice of basis of* $V(\mathbf{O})$.

Proof. The coordinates of a vector with respect to two different bases differ by a matrix with entries in **O**. The argument in the proof of Lemma 4 shows that the norms with respect to two different bases are equal. □

Lemma 6. *The norm* $\|\cdot\|$ *satisfies the* ultrametric triangle inequality, *i.e., for vectors* **x**, **y** *in* $V(F_\bullet)$,

$$\|\mathbf{x}+\mathbf{y}\| \le \sup\{\|\mathbf{x}\|, \|\mathbf{y}\|\}.$$

Proof. Write $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$ and $\mathbf{y} = y_1\mathbf{e}_1 + \cdots + y_n\mathbf{e}_n$. Then

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\| &= \sup\{|x_1+y_1|, \ldots, |x_n+y_n|\} \\
&\le \sup\{\sup\{|x_1|,|y_1|\}, \ldots, \sup\{|x_n|,|y_n|\}\} \\
&= \sup\{|x_1|,|y_1|,\ldots,|x_n|,|y_n|\} \\
&= \sup\{\|\mathbf{x}\|, \|\mathbf{y}\|\}.
\end{aligned}$$

□

Lemma 7. *For a scalar* $\lambda \in F_\bullet$ *and a vector* $\mathbf{x} \in V(F_\bullet)$,

$$\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|.$$

Lemma 8. *If* $g \in GL(V(F_\bullet))$, *then there is a constant* $C_g > 0$, *such that for any vector* $\mathbf{x} \in V(F_\bullet)$,

$$\|\mathbf{x}g\| \le C_g\|\mathbf{x}\|.$$

Proof. Suppose that g has matrix (g_{ij}), and $\mathbf{x}$ has coordinates $(x_1,\dots,x_n)$ with respect to the basis $\mathbf{e}_1,\dots,\mathbf{e}_n$. Then

$$\begin{aligned}\|\mathbf{x}g\| &= \sup\left\{\left|\sum_{i=1}^n x_i g_{i1}\right|,\dots,\left|\sum_{i=1}^n x_i g_{in}\right|\right\}\\ &\leq \sup_{1\leq j\leq n}\sup_{1\leq i\leq n}|g_{ij}|\|\mathbf{x}\|.\end{aligned}$$

Therefore, let

$$C_g = \sup_{1\leq j\leq n}\sup_{1\leq i\leq n}|g_{ij}|.$$

□

Lemma 9. *If* $\mathbf{x}\in V(R)$ *is a non-zero vector then* $\|\mathbf{x}\|\geq 1$.

Proof. By Corollary 5, we may assume that the elements $\mathbf{e}_i$ of a basis used to define $\|\cdot\|$ lie in $V(\mathbf{F}_q)$. Then at least one coordinate of $\mathbf{x}$ is non-zero in R. But any non-zero element in R has norm at least one. Therefore, $\|\mathbf{x}\|\geq 1$. □

PROPOSITION 10.

For any non-zero vector $\mathbf{x}\in V(\mathbf{F}_q)$ *and any* $g\in GL(V(F_\bullet))$, *there is a positive constant* E *such that for all* $\gamma\in GL(V(R))$,

$$\|\mathbf{x}\gamma g\|\geq E.$$

Consequently, for any subset S *of* $GL(V(R))$, *the set* $\{\|\mathbf{x}sg\| : s\in S\}$ *has a positive minimal element.*

Proof. Applying Lemma 8 to g^{-1}, and Lemma 9 to $\mathbf{x}\gamma$ (which lies in $V(R)$), we have

$$\|\mathbf{x}\gamma g\|\geq C_{g^{-1}}\|\mathbf{x}\gamma\|\geq C_{g^{-1}}>0.$$

The second part of the assertion follows by noting that the values taken by the norm $\|\cdot\|$ are of the form q^j, where j is an integer. □

3. Fundamental representations

Let $\alpha_1,\dots,\alpha_r$ be the simple roots with respect to B in the root system $\Phi(G,T)$ of G with respect to T. Let $W=N_G(T)/T$ be the Weyl group of G with respect to T. To each simple root α_i, we associate an element s_i of order two in W in the usual way.

Given a subset D of $\{1,\dots,r\}$, let W_D denote the subgroup of W generated by $\{s_j|j\in D\}$, and let P_D denote the parabolic subgroup BW_DB of G containing B. This group has a Levi decomposition

$$P_D = L_D U_D,$$

where L_D is a reductive group of rank $|D|$ and U_D is the unipotent radical of P_D. $L_D\cap B$ is a Borel subgroup for L_D containing the split torus T. The set of simple roots of L_D with respect to $L_D\cap B$ is $\{\alpha_j|j\in D\}$. Denote by P_i (resp., L_i, U_i) the parabolic subgroup (resp., Levi subgroup, unipotent subgroup) corresponding to the set $\{1,\dots,i-1,i+1,\dots,r\}$. These are the maximal proper parabolic subgroups of G containing B.

Theorem 11. **[2]** *There exist irreducible finite dimensional representations* (ρ_i, V_i) *of* G, *vectors* $\mathbf{v}_i \in V_i(\mathbf{F}_q)$ *that are unique up to scaling, and characters* $\Delta_i : P_i \to \mathbf{G}_m$, *for* $i = 1, \ldots, r$ *all defined over* $\mathbf{F}_q$, *such that*

1. P_i *is the stabilizer of the line generated by* $\mathbf{v}_i$ *and* $\mathbf{v}_i \rho_i(p) = \Delta_i(p)\mathbf{v}_i$ *for each* $p \in P_i$ *for* $i = 1, \ldots, r$.
2. *The restrictions* μ_i *to* T *of* Δ_i*s are antidominant weights of* T *with respect to* B, *which generate* $X^*(T) \otimes \mathbf{Q}$ *as a vector space over the rational numbers.*

4. Ordering by roots

Lemma 12. *Let* L *be a Levi subgroup of* G *associated to a parabolic subgroup* P *containing* B. *Then there is a canonical surjection*

$$G(F_\bullet)/G(\mathbf{O}) \xrightarrow{\Phi_L^G} L(F_\bullet)/L(\mathbf{O}).$$

If $Q = MN$ *is a parabolic subgroup of* G *containing* B *and contained in* P, *then* M *is a Levi subgroup for* L *corresponding to the parabolic subgroup* $L \cap Q$ *of* L, *and* $\Phi_M^L \circ \Phi_L^G = \Phi_M^G$.

Proof. Given $g \in G(F_\bullet)$, we may use the Iwasawa decomposition to write $g = luk$, where $l \in L(F_\bullet)$, $u \in U(F_\bullet)$ and $k \in G(\mathbf{O})$. Moreover, if $g = l'u'k'$ is another such decomposition, then, setting $l_0 = l'^{-1}l$ and $k_0 = k'k^{-1}$,

$$u'^{-1} l_0 u = k_0 \in G(\mathbf{O}).$$

On the other hand,

$$k_0 = u'^{-1} l_0 u = l_0 l_0^{-1} u'^{-1} l_0 u.$$

Since L normalizes U, $l_0^{-1} u'^{-1} l_0 \in U(F_\bullet)$, and hence, setting $u_0 = l_0^{-1} u'^{-1} l_0 u \in U(F_\bullet)$,

$$l_0 = k_0 u_0 \in G(\mathbf{O}) U(F_\bullet) \cap L(F_\bullet).$$

Therefore $l_0 u_0^{-1} = k_0 \in G(\mathbf{O}) \cap P(F_\bullet) = P(\mathbf{O})$, so that $l_0 \in L(\mathbf{O})$. This shows that $luk \mapsto l$ induces a well defined map $\Phi_L^G : G(F_\bullet)/G(\mathbf{O}) \to L(F_\bullet)/L(\mathbf{O})$. It is clear that this map is surjective. To see that $\Phi_M^L \circ \Phi_L^G = \Phi_M^G$, note that we may write $g = muk$ with $m \in M(F_\bullet)$, $u \in N(F_\bullet)$ and $k \in G(\mathbf{O})$. But $N(F_\bullet) = (N(F_\bullet) \cap L(F_\bullet))U(F_\bullet)$, so we may write $u = u_1 u_2$, where $u_1 \in N(F_\bullet) \cap L(F_\bullet)$ and $u_2 \in U(F_\bullet)$. Therefore, we see that $mM(\mathbf{O}) = \Phi_M^L(mu_1) = \Phi_M^G(g)$. □

In the sequel we denote Φ_T^G simply by Φ. Define

$$\Omega_G := \{g \in G(F_\bullet) \;:\; |\alpha_i \circ \Phi(g)| \geq 1 \text{ for } i = 1, \ldots, r\}. \tag{2}$$

PROPOSITION 14.

$G(F_\bullet) = \Gamma \Omega_G$.

Proof.

The rank one case (following [15]): Here G has one simple root α_1, and one fundamental representation (ρ_1, V_1) and a vector $\mathbf{v}_1 \in V_1(\mathbf{F}_q)$ such that for any element p in the parabolic subgroup $B = TN$, where N is the unipotent radical of B,

$$\mathbf{v}_1\rho_1(b) = \Delta_1(b)\mathbf{v}_1, \tag{3}$$

where the character $\Delta_1 : B \mapsto \mathbf{G}_m$ (defined over $\mathbf{F}_q$) restricts to an anti-dominant weight μ_1 on the maximal split torus T. Let $g \in G(F_\bullet)$. We wish to show that $g \in \Gamma\Omega_G$. To this end, by Proposition 10, and by replacing g, if necessary by an appropriate element of Γg, we may assume that g has the property that

$$\|\mathbf{v}_1\rho_1(\gamma g)\| \geq \|\mathbf{v}_1\rho_1(g)\| \quad \text{for all } \gamma \in \Gamma. \tag{4}$$

Write $g = tnk$, where $t \in T(F_\bullet)$, $n \in N(F_\bullet)$ and $k \in G(\mathbf{O})$. By Theorem 11 and Lemma 4,

$$\|\mathbf{v}_1\rho(g)\| = |\Delta_1(t)|\|\mathbf{v}_1\| = |\mu_1(t)|. \tag{5}$$

Fix an isomorphism $u_{\alpha_1} : \mathbf{G}_a \to N$ defined over $\mathbf{F}_q$, and let $x \in F_\bullet$ be such that $n = u_{\alpha_1}(x)$. Choose σ in the nontrivial $T(\mathbf{F}_q)$-coset of $N_G T(\mathbf{F}_q)$. Note that if $S \in R$, then $\sigma u_{\alpha_1}(S) \in \Gamma$, therefore, using Proposition 10,

$$\begin{aligned}
|\mu_1(t)| &= \|\mathbf{v}_1\rho_1(g)\| \\
&\leq \|\mathbf{v}_1\rho_1(\sigma u_{\alpha_1}(S) t u_{\alpha_1}(x))\| \\
&= \|\mathbf{v}_1\rho_1({}^{\sigma}t\sigma u_{\alpha_1}(\alpha_1(t)^{-1}(S + \alpha_1(t)x)))\| \\
&= |\mu_1(t)|^{-1}\|\mathbf{v}_1\rho_1(u_{-\alpha_1}(\alpha(t)^{-1}S + x))\|.
\end{aligned}$$

Here $u_{-\alpha_1} = \sigma u_{\alpha_1}\sigma^{-1}$, and its image is the root subgroup for $-\alpha_1$. The element $u_{-\alpha_1}(\alpha(t)^{-1}S + x)$ lies in the derived group of G which is isomorphic to either SL_2 or PGL_2 in the rank one case. When the derived group of G is isomorphic to SL_2, we may take V_1 to be the right action of SL_2 on the space of 1×2-matrices by right multiplication. One may take the torus T to consist of diagonal matrices in SL_2, B the upper triangular matrices in SL_2 and $\mathbf{v}_1$ to be the vector $(0,1)$. Calculating with matrices, one may verify that

$$\|\mathbf{v}_1\rho_1(u_{-\alpha_1}(\alpha(t)^{-1}S + x))\| \leq \sup\{1, |\alpha(t)^{-1}S + x|\}.$$

Therefore,

$$\sup\{1, |\alpha_1(t)^{-1}S + x|\} \geq |\mu_1(t)|^2. \tag{6}$$

Choose S in R such that $|S + \alpha(t)x| < 1$. Then $|\alpha_1(t)^{-1}S + x| < |\alpha_1(t)|^{-1}$. Suppose that $|\alpha_1(t)^{-1}S + x| \geq |\mu_1(t)|^2$. Then $|\alpha_1(t)|^{-1} > |\mu_1(t)|^2$. This is impossible, since $\alpha_1(t)^{-1} = \mu_1(t)^2$. It follows that $|\alpha_1(t)^{-1}S + x| < |\mu_1(t)|^2$. Therefore, (6) can hold only if $1 \geq |\mu_1(t)|^2$, which is the same as $|\alpha_1(t)| \geq 1$. This completes the proof of Proposition 14 when the derived group of G is isomorphic to SL_2.

When the derived group of G is isomorphic to PGL_2, then G is the product of its centre with PGL_2. Therefore, the assertion of Proposition 14 for G follows from that for PGL_2. However, the assertion for PGL_2 follows easily from that for GL_2. The derived group of

GL_2 is SL_2, hence the proposition holds for GL_2 by the argument in the previous paragraph, completing the proof of Proposition 14 in the rank one case.

The general case: Let G be a group of rank r, and $g \in G(F_\bullet)$. By modifying g on the left by an element of Γ, we may, for the purposes of this proof, assume, using the second assertion of Proposition 10, that

$$\|\mathbf{v}_1\rho_1(g)\| \leq \|\mathbf{v}_1\rho_1(\gamma g)\| \quad \text{for all } \gamma \in \Gamma. \tag{7}$$

Note that if $\gamma \in P_1(F_\bullet) \cap \Gamma$, then $\mathbf{v}_1\rho_1(\gamma g) = \Delta_1(\gamma)\mathbf{v}_1\rho_1(g)$. Since $\Delta_1(\gamma) \in \mathbf{F}_q[\pi^{-1}]^\times$, $|\Delta_1(\gamma)| = 1$. Therefore, $\|\mathbf{v}_1\rho_1(\gamma g)\| = \|\Delta_1(\gamma)\mathbf{v}_1\rho_1(g)\|$. We may use the second assertion of Proposition 10 again, to assume, for the purposes of this proof, that

$$\|\mathbf{v}_2\rho_2(g)\| \leq \|\mathbf{v}_2\rho_2(\gamma g)\| \quad \text{for all } \gamma \in \Gamma \cap P_1(F_\bullet) \tag{8}$$

while preserving (7). Continuing in this manner, we may assume that

$$\|\mathbf{v}_j\rho_j(g)\| \leq \|\mathbf{v}_j\rho_j(\gamma g)\| \quad \text{for all } \gamma \in \Gamma \cap P_1(F) \cap \ldots \cap P_{j-1}(F), \tag{9}$$

for $j = 1, \ldots, r$. Therefore, it suffices to prove the following:

Lemma 22. *If an element $g \in G(F_\bullet)$ satisfies the inequalities* (9) *for each integer $1 \leq j \leq r$, then $g \in \Omega_G$.*

The proof of Proposition 14 in the rank one case shows that Lemma 22 is true when G is of semisimple rank one. We prove it in general assuming the validity of Theorem 2 in the rank one case.

Suppose that g satisfies the inequalities (9) for each $1 \leq j \leq r$. Write $g = bk$, with $b \in B(F_\bullet)$ and $k \in G(\mathbf{O})$. Then b can be written as lu, where $l \in L_{\{i\}}(F_\bullet) \cap B(F_\bullet)$ and $u \in U_{\{i\}}(F_\bullet)$. Since $U_{\{i\}}$ fixes $\mathbf{v}_i$, the inequalities (9) imply that

$$\|\mathbf{v}_i\rho_i(l)\| \leq \|\mathbf{v}_i\rho_i(\gamma l)\| \quad \text{for all } \gamma \in L_{\{i\}}(R). \tag{10}$$

From the rank one case, $l = \gamma\pi^\eta k$ for some $\gamma \in L_{\{i\}}(R)$, $k \in L_{\{i\}}(\mathbf{O})$ and $\eta \in X_*(T)$ such that $|\alpha_i(\pi^\eta)| \geq 1$. $\rho_i(\gamma)$ maps $\mathbf{v}_i$ into $V(R)$. From Lemma 24 it follows that

$$\|\mathbf{v}_i\rho_i(l)\| \geq \|\mathbf{v}_i\rho_i(\pi^\eta)\|.$$

Equation (10) implies that the above must be an equality. This forces $\gamma \in L_{\{i\}}(R) \cap P_i(R)$, and hence also $k \in L_{\{i\}}(\mathbf{O}) \cap P_i(\mathbf{O})$. Write $b = tn$ with $t \in T(F_\bullet)$ and $n \in N(F_\bullet)$. Then viewing α_i as a rational character of $B(F_\bullet)$ that is trivial on $N(F_\bullet)$, we have

$$|\alpha_i(t)| = |\alpha_i(l)| = |\alpha_i(\pi^\eta)| \geq 1.$$

Repeating this argument for each i completes the proof of Lemma 22. $\square$

5. Local reduction theory

In order to prove the existence part of Theorem 2, it suffices to show that every element g in Ω_G may be written as $g = \gamma\pi^\eta k$, where $\gamma \in \Gamma$, $\eta \in X_*(T)$ is antidominant and $k \in G(\mathbf{O})$. To this end, we may assume (using the Iwasawa decomposition) that we are given

$g \in \Omega_G$, with $g = tn$, with $t \in T(F_\bullet)$ and $n \in N(F_\bullet)$. Since g, and hence t, is in Ω_G, $|\alpha_i(t)| \geq 1$, so that $\alpha_i(t)^{-1} \in \mathbf{O}$, for $i = 1, \ldots, r$. For each root $\alpha \in \Phi(G,T)$, let U_α denote the corresponding root subgroup. Fix an isomorphism $u_\alpha : \mathbf{G}_a \to U_\alpha$ defined over $\mathbf{F}_q$. Then for $x \in F_\bullet$, we have

$$tu_\alpha(x) = (tu_\alpha(x)t^{-1})t = u_\alpha(\alpha(t)x)t.$$

Therefore, if we write $\alpha(t)x = P + h$, where $P \in R$ and $h \in \mathbf{O}$, then

$$tu_\alpha(x) = tu_\alpha(\alpha(t)^{-1}P)u_\alpha(\alpha(t)^{-1}h) = u_\alpha(P)tu_\alpha(\alpha(t)^{-1}h).$$

Given two positive roots α and β, the commutator $[U_\alpha, U_\beta]$ is contained in the product of root subgroups $U_{\alpha'}$ where the α' are roots which can be written as positive linear combinations of α and β and are distinct from either α or β. Moreover, we may enumerate the positive roots as $\beta_1, \beta_2, \ldots$ so that if $j > i$, then β_i cannot be written as a sum of β_j and any other positive roots.

Write n as $\prod_i u_{\beta_i}(x_i)$. Then

$$tn = tu_{\beta_1}(x_1)\prod_{i>1} u_{\beta_i}(x_i).$$

If we write $\beta_1(t)x_1 = P_1 + h_1$, where $P_1 \in \mathbf{F}_q[\pi^{-1}]$ and $h \in \mathbf{O}$, then

$$tn = u_{\beta_1}(P_1)tu_{\beta_1}(\beta_1(t)^{-1}h_1)\prod_{i>1} u_{\beta_i}(x_i).$$

Since $u_{\beta_1}(P_1) \in \Gamma$, $\beta_1(t)^{-1} \in \mathbf{O}$, and the image of u_{β_1} normalizes all the subsequent root subgroups whose elements appear in the above expression, we may assume for the purpose of proving Theorem 2, that

$$tn = t\prod_{i>1} u_{\beta_i}(x_i'),$$

for $x_i' \in F_\bullet$. We may continue in this manner to reduce tn to t. It is then easy to see (using the decomposition $F_\bullet^\times = \pi^{\mathbf{Z}}\mathbf{O}^\times$) that t may be replaced by π^η for $\eta \in X_*(T)$. Since $|\alpha_i(\pi^\eta)| \geq 1$, it follows that η is antidominant, proving the existence part of Theorem 2.

We now prove the uniqueness part of Theorem 2. In order to do this, it suffices to show that if η and ν are two dominant co-weights, and $\pi^\nu = \gamma\pi^\eta k$ for some $\gamma \in \Gamma$ and $k \in G(\mathbf{O})$, then $\nu = \eta$. Since the weights $\mu_1, \ldots, \mu_r$ corresponding to the fundamental representations in Theorem 11 generate the vector space $X^*(T) \otimes \mathbf{Q}$, it suffices to show that $\langle \mu_i, \nu \rangle = \langle \mu_i, \eta \rangle$ for each i. In order to do this, we need the following:

Lemma 24. *For any non-zero vector* $\mathbf{v} \in V_i(F_\bullet)$ *and any antidominant co-weight* $\mu \in X_*(T)$,

$$\frac{\|\mathbf{v}\rho_i(\pi^\mu)\|}{\|\mathbf{v}\|} \geq \frac{\|\mathbf{v}_i\rho_i(\pi^\mu)\|}{\|\mathbf{v}_i\|}.$$

Proof. Since T is an $\mathbf{F}_q$-split torus and ρ_i is defined over $\mathbf{F}_q$, V has a decomposition (over $\mathbf{F}_q$) into root subspaces

$$V = \bigoplus_\lambda V_\lambda,$$

where T acts on V_λ by the character $\lambda : T \to \mathbf{G}_m$. It is easy to see that μ_i is the lowest weight of T occurring in (ρ_i, V_i), so that $\langle \mu_i, \mu \rangle \geq \langle \lambda, \mu \rangle$ for any weight λ of T occurring in (ρ_i, V_i) and any antidominant co-weight μ. Given any vector $\mathbf{v} \in V(F_\bullet)$, we may write

$$\mathbf{v} = \sum x_j \mathbf{u}_j,$$

where $x_j \in F_\bullet$ and $\mathbf{u}_j \in V_{\lambda_j}(\mathbf{F}_q)$ for each j and the λ_js are not necessarily distinct. Thus

$$\begin{aligned}
\|\mathbf{v}\rho_i(\pi^\mu)\| &= \left\|\sum \lambda_j(\pi^\mu) x_j \mathbf{u}_j\right\| \\
&= \sup_j \{|\lambda_j(\pi^\mu) x_j|\} \\
&= \sup_j \{q^{-\langle \lambda_j, \mu \rangle} |x_j|\} \\
&\geq q^{-\langle \mu_i, \mu \rangle} \sup_j \{|x_j|\} \\
&= \|\mathbf{v}_i \rho_i(\pi^\mu)\| \, \|\mathbf{v}\|.
\end{aligned}$$

Since $\|\mathbf{v}_i\| = 1$, this completes the proof of Lemma 24. $\square$

Lemma 24 allows us to compare $\langle \mu_i, \nu \rangle$ and $\langle \mu_i, \eta \rangle$:

$$\begin{aligned}
q^{-\langle \mu_i, \eta \rangle} &= \frac{\|\mathbf{v}_i \rho_i(\pi^\eta)\|}{\|\mathbf{v}_i\|} \\
&\leq \frac{\|\mathbf{v}_i \rho_i(\gamma\pi^\eta)\|}{\|\mathbf{v}_i \rho_i(\gamma)\|} \\
&\leq \frac{\|\mathbf{v}_i \rho_i(\gamma\pi^\eta)\|}{\|\mathbf{v}_i\|} \\
&= \frac{\|\mathbf{v}_i \rho_1(\pi^\nu)\|}{\|\mathbf{v}_i\|} \\
&= q^{-\langle \mu_i, \nu \rangle}.
\end{aligned}$$

The first inequality is Lemma 24 applied to $\mathbf{v} = \mathbf{v}_i \rho_i(\gamma)$. The second inequality follows from Lemma 9 with $\mathbf{x} = \mathbf{v}_i \rho_i(\gamma)$. Interchanging the roles of η and ν in the above arguments shows that $\langle \mu_i, \eta \rangle = \langle \mu_i, \nu \rangle$ for each i. This completes the proof of the uniqueness part of the assertion of Theorem 2.

6. Global reduction theory

If $g = (g_v)_v$ is an element of $G(\mathbf{A})$ then, since $g_v \in G(\mathbf{O}_v)$ for all but finitely many places v of F, we may assume, for the purpose of proving Theorem 1 that g is a finite product $g = g_\infty g_{v_1} g_{v_2} \dots g_{v_k}$, with $g_\infty \in G(F_\infty)$ and $g_{v_j} \in G(F_{v_j})$, $v_j \neq \infty$, for $1 \leq j \leq k$. By Theorem 2, there is a decomposition

$$g_{v_k} = \gamma_k \pi_{v_k}^{\eta_k} \kappa_k,$$

where $\gamma_k \in G(\mathbf{F}_q[\pi_{v_k}^{-1}])$, $\eta_k \in X_*(T)$, and $\kappa_k \in G(\mathbf{O}_{v_k})$. Now γ_k and $\pi_{v_k}^{\eta_k}$ are both contained in $G(F)$ and in $G(\mathbf{O}_v)$ for all $v \neq \infty$. Therefore, by multiplying g on the left by $\pi_{v_k}^{-\eta_k}\gamma^{-1}$ we get an element of the subset

$$G(F_\infty) \times \prod_{j=1}^{k-1} G(F_{v_j}) \times \prod_{\text{all other } v} G(\mathbf{O}_v)$$

of $G(\mathbf{A})$.

We have now reduced g to an element with non-trivial entries only at most $k-1$ places and ∞. We may continue in this manner until the entries at all places except ∞ are trivial. Finally, the use of Theorem 2 to $v = \infty$ gives us a representative each double coset of type asserted by Theorem 1.

The uniqueness part of the theorem follows from the corresponding assertion in the local situation, because two elements g and h of $G(F_\infty)$ lie in the same double coset if and only if $g = \gamma hk$, with $\gamma \in G(\mathbf{F}_q[t])$ and $k \in G(\mathbf{O}_\infty)$.

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